Indukované zobrazení, diffeomorfismy, toky a Lieova derivace

Indukované zobrazení na tenzorech

Diferenciál zobrazení

Indukované zobrazení na polích

Tok a jeho generátor

Lieova derivace

Interpretace Lieovy závorky

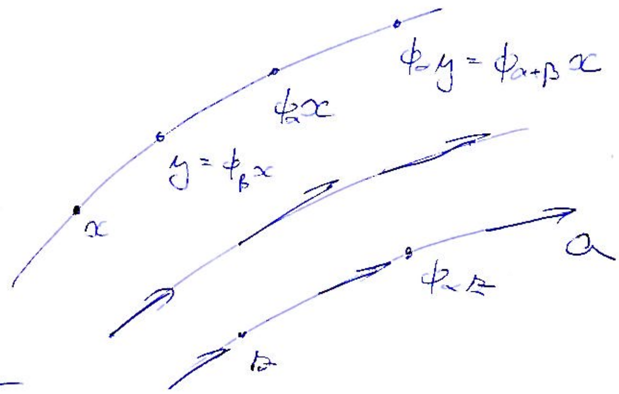
Tok a jeho generátor

Def tok na M, též 1-param. grupa diffeo

$$\phi_\alpha \in \mathrm{Diff}\, M \qquad \alpha \in \mathbb{R}$$

$$\phi_\alpha \circ \phi_\beta = \phi_{\alpha+\beta}$$

$$\phi_0 = id \qquad \phi_{-\alpha} = \phi_\alpha^{-1}$$



všechny body na orbitě $\phi_\alpha x$ se pohybují po stejné orbitě

Def generátor toku

a je generátor ϕ_α $\equiv$

$$a|_x = \frac{D}{d\alpha} \phi_\alpha x \Big|_{\alpha=0} \qquad \phi_\alpha = \mathrm{diff}_\alpha[a]$$

Tk: tok a jeho generátor se vzájemně jedn. určují

tj gen a definuje jedn. ϕ_α

$\Leftarrow$ věty o řešení dif. rov. 1. řádu

Pozn:

$$\phi_{\alpha *} a = a$$

$$\Leftarrow (\phi_{\alpha *} a)(\phi_\alpha x) = \phi_{\alpha *}(a(x)) = \phi_{\alpha *}\left(\frac{D}{d\varepsilon} \phi_\varepsilon x \Big|_{\varepsilon=0}\right) = \frac{D}{d\varepsilon} \phi_\varepsilon \phi_\alpha x \Big|_{\varepsilon=0} = a(\phi_\alpha x)$$

Lieova derivace

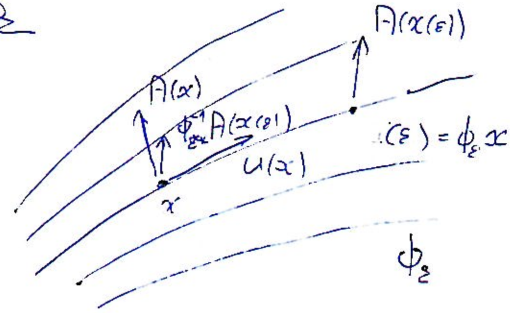
chceme charakteriz. derivaci tenz pole A ve směru u
formální výraz

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(A(x+\varepsilon u) - A(x)\right)$$

malorodiné křivka ve směru u

$\mathbb{T}_{x(\varepsilon)}M$ $\qquad$ $\mathbb{T}_x M$ – nelze porovnávat

potřeba přenést tenzory do stejného bodu
nelze provést pouze pomocí specifikace směru $u\in\mathbb{T}_x M$
potřeba dodatečné struktury – tok



Def: Lieova derivace

$$\mathcal{L}_u A\big|_x = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\phi_{-\varepsilon *}(A(\phi_\varepsilon x)) - A(x)\right)$$

kde ϕ_α je tok generovaný u

alternativní zápisy

$$\mathcal{L}_u A\big|_x = \frac{d}{d\varepsilon}\phi_{-\varepsilon *}(A(\phi_\varepsilon x))\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\phi^*_\varepsilon(A(\phi_\varepsilon x))\Big|_{\varepsilon=0}$$

$$\mathcal{L}_u A = -\frac{d}{d\varepsilon}\phi_{\varepsilon *}A\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\phi^*_\varepsilon A\Big|_{\varepsilon=0}$$

$$\Leftarrow \phi_{-\varepsilon *}(A(\phi_\varepsilon x)) = (\phi_{-\varepsilon *}A)(x) = (\phi^*_\varepsilon A)(x)$$

Tk. ϕ_α tok s generátorem a
indukované zobr ϕ^* na tenz. polích splňuje

$$\phi^*_0 = id \qquad \phi^*_\alpha\circ\phi^*_\beta = \phi^*_{\alpha+\beta} \qquad \frac{d}{d\alpha}\phi^*_\alpha\Big|_{\alpha=0} = \mathcal{L}_a$$

proto splňuje i

$$\frac{d}{d\alpha}\phi^*_\alpha = \phi^*_\alpha\mathcal{L}_a \quad\Rightarrow\quad \phi^*_\alpha = \exp[\alpha\mathcal{L}_a]$$

důk:

$$\phi^*_\alpha\mathcal{L}_a A = \phi^*_\alpha\frac{d}{d\varepsilon}\phi^*_\varepsilon A\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\phi^*_{\alpha+\varepsilon}A\Big|_{\varepsilon=0} = \frac{d}{d\alpha}\phi^*_\alpha A$$

vlastnosti Lieovy derivace

$$\mathcal{L}_u(A + rB) = \mathcal{L}_u A + r\mathcal{L}_u B \qquad r \in \mathbb{R}$$

$$\mathcal{L}_u(A \otimes B) = (\mathcal{L}_u A) \otimes B + A \otimes (\mathcal{L}_u B)$$

$$\mathcal{L}_u \mathsf{C} A = \mathsf{C}\, \mathcal{L}_u A$$

$$\mathcal{L}_u f = u \cdot df$$

$$\mathcal{L}_u df = d\mathcal{L}_u f$$

$$\mathcal{L}_u(a \cdot \omega) = (\mathcal{L}_u a) \cdot \omega + a \cdot \mathcal{L}_u \omega$$

$$\mathcal{L}_u v = [u, v]$$

důkazy

přímočaře plyne z vlastností ϕ^*

– komutace s $+, \otimes, \mathsf{C}, d$

a převedení na obyčejnou der. $\frac{d}{d\varepsilon} \phi^*_\varepsilon A \big|_{\varepsilon = 0}$

$\mathcal{L}_a f = a[f]$ – přímo z def.

$$\mathcal{L}_u(v \cdot df) = u \cdot d(v \cdot df) =$$

$$= (\mathcal{L}_u v) \cdot df + v \cdot \mathcal{L}_a df = (\mathcal{L}_u v) \cdot df + \cancel{v} \cdot d(u \cdot df)$$

$$\Rightarrow (\mathcal{L}_u v) \cdot df = u[v[f]] - v[u[f]] = [u, v][f]$$

Th: Lieova der. $\mathcal{L}_a$ je tenz. derivace

na $\mathcal{F}M$ určena vekt. polem a

na TM určena Lieovou závorkou s a

!! jednoznačně fixovaná na tenz. polích

Tl: linearita (nad $\mathbb{R}$) ve směru

$$\mathcal{L}_{u+rv} = \mathcal{L}_u + r\mathcal{L}_v \qquad r \in \mathbb{R}$$

důk:

$\mathbb{L} = \mathcal{L}_{u+rv} - \mathcal{L}_u - r\mathcal{L}_v$ je tenz. der. splňující

$\mathbb{L} f = (u+rv)[f] - u[f] - rv[f] = 0 \qquad \mathbb{L} w = [(u+rv), w] - [u, w] - r[v, w] = 0$

$\Rightarrow \mathbb{L} = 0$ c.b.d.

Tl:

$$\mathcal{L}_{[u,v]} = \mathcal{L}_u\mathcal{L}_v - \mathcal{L}_v\mathcal{L}_u$$

důk:

$\mathbb{L} = \mathcal{L}_{[u,v]} - (\mathcal{L}_u\mathcal{L}_v - \mathcal{L}_v\mathcal{L}_u)$ je tenz. der.

linearita zřejmá

Leibnitz

$[\mathcal{L}_u\mathcal{L}_v - \mathcal{L}_v\mathcal{L}_u](AB) = \mathcal{L}_u((\mathcal{L}_v A)B + A\mathcal{L}_v B) - (u \leftrightarrow v)$

$= (\mathcal{L}_u\mathcal{L}_v A)B + A(\mathcal{L}_u\mathcal{L}_v B) + (\mathcal{L}_v A)(\mathcal{L}_u B) + (\mathcal{L}_u A)(\mathcal{L}_v B) - (u \leftrightarrow v)$

$= ([\mathcal{L}_u\mathcal{L}_v - \mathcal{L}_v\mathcal{L}_u]A)B + A([\mathcal{L}_u\mathcal{L}_v - \mathcal{L}_v\mathcal{L}_u]B)$

$\Rightarrow \mathbb{L}(AB) = (\mathbb{L}A)B + A(\mathbb{L}B)$

na funkcích $\mathbb{L} f = [u,v][f] - (u[v[f]] - v[u[f]]) = 0$

na vekt. $\mathbb{L} w = [[u,v],w] - [u,[v,w]] - [v[u,w]] = 0$ J.I.

$\Rightarrow \mathbb{L} = 0$ c.b.d.

Tl: $\mathcal{L}_w[u,v] = [\mathcal{L}_w u, v] + [u, \mathcal{L}_w v]$

důk:

$[w,[u,v]] = [[w,u],v] + [u,[w,v]] \Leftarrow$ J.I. c.b.d.

Tl: souřadnicové vyjádření

$$\left(\mathcal{L}_{\frac{\partial}{\partial x^1}} A\right)^{a\ldots}_{b\ldots} = A^{a\ldots}_{b\ldots,1}$$

kde $A^{a\ldots}_{b\ldots}$ jsou kanon. vůči x^a

důk:

$\mathcal{L}_{\frac{\partial}{\partial x^1}} \frac{\partial}{\partial x^a} = 0 \qquad \mathcal{L}_{\frac{\partial}{\partial x^1}} dx^a = d\frac{\partial x^a}{\partial x^1} = 0 \qquad \mathcal{L}_{\frac{\partial}{\partial x^1}}(A^{a\ldots}_{b\ldots}) = A^{a\ldots}_{b\ldots,1}$

Lieova derivace forem

Lieova der. na formách splňuje

$$\mathcal{L}_a : \Omega^p M \to \Omega^p M$$

$$\mathcal{L}_a(\omega + \sigma) = \mathcal{L}_a\omega + \mathcal{L}_a\sigma$$

$$\mathcal{L}_a(\omega\wedge\sigma) = (\mathcal{L}_a\omega)\wedge\sigma + \omega\wedge(\mathcal{L}_a\sigma)$$

$$\mathcal{L}_a f = a\cdot df \qquad\qquad f \in \mathcal{F}M = \Omega^0 M$$

$$\mathcal{L}_a df = d\mathcal{L}_a f$$

Tyto vlastnosti určují operaci na formách jednozn.

důk:

Leibnitz plyne z Leibnitze pro $\otimes$ a vztahu

$\omega\wedge\sigma = \bigcirc\,\mathcal{A}(\omega\otimes\sigma)$ + linearita

jednozn. plyne z rozkladu obecného ω na kompon.

Cartanova identita

$$\mathcal{L}_a\omega = a\cdot d\omega + d(a\cdot\omega)$$

$$\mathcal{L}_a = \iota_a d + d\,\iota_a \qquad \text{(homotopy id)}$$

důkaz 1: indukcí přes stupeň

$p=0$ $\quad \mathcal{L}_a f = a\cdot df + d(\cancel{a\cdot f^0})$ ✓

$p\to p+1$ $\quad \alpha\in\Omega^p M \quad \omega\in\Omega^{p+1}$ - suma členů typu $df\wedge\alpha$

$$\mathcal{L}_a(df\wedge\alpha) = (d\mathcal{L}_a f)\wedge\alpha + df\wedge\mathcal{L}_a\alpha =$$

$$= (d(a\cdot df))\wedge\alpha \qquad + df\wedge(a\cdot d\alpha) \qquad + df\wedge d(a\cdot\alpha)$$

$$= d((a\cdot df)\wedge\alpha) - \cancel{(a\cdot df)\,d\alpha} - a\cdot(df\wedge d\alpha) + \cancel{(a\cdot df)\,d\alpha} - d(df\wedge(a\cdot\alpha))$$

$$= d(a\cdot(df\wedge\alpha)) + \cancel{d(df\wedge(a\cdot\alpha))} + a\cdot d(df\wedge\alpha) - \cancel{d(df\wedge(a\cdot\alpha))}$$

$$= a\cdot d(df\wedge\alpha) + d(a\cdot(df\wedge\alpha)) \qquad \text{c.b.d.}$$

důkaz 2: operace $a\cdot d + d\,\iota_a$ splňuje vlastnosti výše

Leibnitz: $a\cdot d(\omega\wedge\sigma) + d(a\cdot(\omega\wedge\sigma)) = a\cdot(d\omega\wedge\sigma + (-1)^p\omega\wedge d\sigma) + d((a\cdot\omega)\wedge\sigma + (-1)^p\omega\wedge(a\cdot\sigma))$

$= (a\cdot d\omega + d(a\cdot\omega))\wedge\sigma + (-1)^{2p}\omega\wedge(a\cdot d\sigma + d(a\cdot\sigma))$

$+ \cancel{(-1)^{p+1}d\omega\wedge(a\cdot\sigma)} + \cancel{(-1)^p(a\cdot\omega)\wedge d\sigma} + \cancel{(-1)^p d\omega\wedge(a\cdot\sigma)} + \cancel{(-1)^{p-1}(a\cdot\omega)\wedge d\sigma}$ $\quad$ c.b.d.

platí

Tk: $\mathcal{L}_a d = d\mathcal{L}_a$ na celé ΛM

důk: $\mathcal{L}_a d\omega = a\cdot d\cancel{d\omega}^{0} + d(a\cdot d\omega) =$

$= d(\mathcal{L}_a\omega - d(a\cdot\omega)) = d\mathcal{L}_a\omega$ c.b.d.

Tk: $\mathcal{L}_a \iota_b - \iota_b\mathcal{L}_a = \iota_{[a,b]}$

důk: $\mathcal{L}_a(b\cdot\omega) = (\mathcal{L}_a b)\cdot\omega + b\cdot(\mathcal{L}_a\omega) =$

$= [a,b]\cdot\omega + \iota_b\mathcal{L}_a\omega$ c.b.d.

Tk: $\mathcal{L}_{fa}\omega = f\mathcal{L}_a\omega + df\wedge(a\cdot\omega)$

důk: $\mathcal{L}_{fa}\omega = fa\cdot d\omega + d(fa\cdot\omega) =$

$= f(a\cdot d\omega + d(a\cdot\omega)) + df\wedge(a\cdot\omega)$

$= f\mathcal{L}_a\omega + df\wedge(a\cdot\omega)$

Interpretace Lieovy závorky

Th: ϕ_α, ψ_β toky s generátory a, b, platí

$$\phi_\alpha \psi_\beta = \psi_\beta \phi_\alpha \iff [a,b] = 0$$

důk:

$\Rightarrow$ definujme $\Xi(\alpha,\beta) = \phi_\alpha \psi_\beta$

orbita $\Xi(\alpha,\beta)x_0$ vytváří 2-di. varietu N_{x_0} se souř. α, β

platí $a|_{N_{x_0}} = \frac{\partial}{\partial\alpha}$ $b|_{N_{x_0}} = \frac{\partial}{\partial\beta}$ $\Rightarrow$ $[a,b]|_{N_{x_0}} = [\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}] = 0$ –

různou volbou x_0 se dostane foliace M podvar. N_x $\Rightarrow$ $[a,b] = 0$

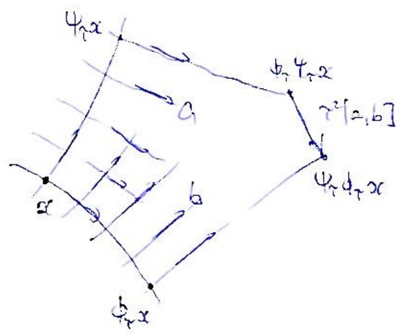
$\Leftarrow$ $\phi_\alpha = \exp[\alpha \mathcal{L}_a]$ $\psi_\beta = \exp[\beta \mathcal{L}_b]$

$[a,b]=0 \Rightarrow [\mathcal{L}_a, \mathcal{L}_b] = 0 \Rightarrow \exp[\alpha\mathcal{L}_a]\exp[\beta\mathcal{L}_b] = \exp[\beta\mathcal{L}_b]\exp[\alpha\mathcal{L}_a] \Rightarrow \phi_\alpha\psi_\beta = \psi_\beta\phi_\alpha$

Th: ϕ_α, ψ_β toky s generátory a, b

$$[\phi_\tau, \psi_\tau]^* f = \tau^2 [a,b]\cdot df + O(\tau^3)$$

důkaz %



Th $a_k \in TU$ $k = 1 \dots K$ U okolí x

$[a_k, a_l] = 0$ na U a_k nezávislé

$\Rightarrow$ bodem x_0 prochází podvarieta N_{x_0} ke které jsou a_k tečné a na ní lze zvolit souř. x^k tak, že $a_k = \frac{\partial}{\partial x^k}$

důk - viz 2D případ

${}^k\phi_\alpha$ tok generovaný a_k

$\Xi(\alpha^1, \dots, \alpha^K) = {}^1\phi_{\alpha^1} \cdots {}^K\phi_{\alpha^K}$ nezávisí na pořadí ${}^k\phi_{\alpha^k}$

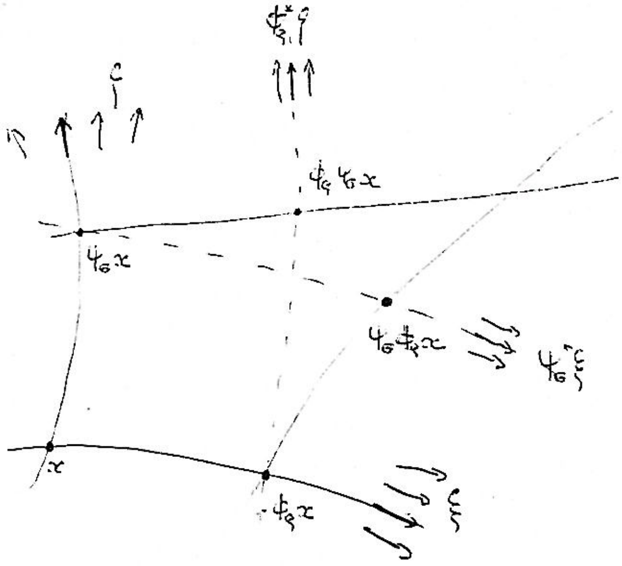
$N_x = \Xi(\alpha^1, \dots, \alpha^K) x_0$ $\alpha^k \in (-\varepsilon, \varepsilon)$ K-di. podvarieta M

$x^k(\Xi(\alpha^a)x_0) = \alpha^k$ → souř. $x = (x^k)$

zřejmě $\frac{\partial}{\partial x^k}\Big|_x = \frac{D}{d\alpha^k} \Xi(\alpha^1 \dots \alpha^k \dots \alpha^K) x_0 \Big|_{\substack{\alpha^k = x^k \\ x^l = x^l(x)}} = a_k|_x$

důsledek

e_k je holonomní báze $\iff$ $[e_k, e_l] = 0$



$$\varphi(\rho,\sigma) = f(\varphi_\rho \psi_\sigma x) \qquad \varphi(0,0) = f(x)$$

$$\psi(\sigma,\rho) = f(\psi_\sigma \varphi_\rho x) \qquad \psi(0,0) = f(x)$$

$$\frac{\partial \varphi}{\partial \rho}(\rho,\sigma) = (\xi \cdot df)\big|_{\varphi_\rho \psi_\sigma x} \qquad \frac{\partial \varphi}{\partial \rho}(0,0) = \xi \cdot df\big|_x$$

$$\frac{\partial \varphi}{\partial \sigma}(\rho,\sigma) = ((\varphi_\rho^* \zeta) \cdot df)\big|_{\varphi_\rho \psi_\sigma x} \qquad \frac{\partial \varphi}{\partial \sigma}(0,0) = \zeta \cdot df\big|_x$$

$$\frac{\partial \psi}{\partial \sigma}(\sigma,\rho) = (\zeta \cdot df)\big|_{\psi_\sigma \varphi_\rho x} \qquad \frac{\partial \psi}{\partial \sigma}(0,0) = \zeta \cdot df\big|_x$$

$$\frac{\partial \psi}{\partial \rho}(\sigma,\rho) = ((\psi_\sigma^* \xi) \cdot df)\big|_{\psi_\sigma \varphi_\rho x} \qquad \frac{\partial \psi}{\partial \rho}(0,0) = \xi \cdot df\big|_x$$

$$\frac{\partial^2 \varphi}{\partial \rho^2}(\rho,\sigma) = (\xi \cdot d(\xi \cdot df))\big|_{\varphi_\rho \psi_\sigma x} \qquad \frac{\partial^2 \varphi}{\partial \rho^2}(0,0) = \xi \cdot d(\xi \cdot df)\big|_x$$

$$\frac{\partial^2 \varphi}{\partial \sigma^2}(\rho,\sigma) = ((\varphi_\rho^* \zeta) \cdot d((\varphi_\rho^* \zeta) \cdot df))\big|_{\varphi_\rho \psi_\sigma x} \qquad \frac{\partial^2 \varphi}{\partial \sigma^2}(0,0) = \zeta \cdot d(\zeta \cdot df)\big|_x$$

$$\frac{\partial^2 \varphi}{\partial \sigma \partial \rho}(\rho,\sigma) = ((\varphi_\rho^* \zeta) \cdot d(\xi \cdot df))\big|_{\varphi_\rho \psi_\sigma x} \qquad \frac{\partial^2 \varphi}{\partial \sigma \partial \rho}(0,0) = \zeta \cdot d(\xi \cdot df)\big|_x$$

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$$\frac{\partial^2 \psi}{\partial \rho \partial \sigma}(\sigma,\rho) = ((\psi_\sigma^* \xi) \cdot d(\zeta \cdot df))\big|_{\psi_\sigma \varphi_\rho x} \qquad \frac{\partial^2 \psi}{\partial \rho \partial \sigma}(0,0) = \xi \cdot d(\zeta \cdot df)\big|_x$$

$$f(\psi_\tau \varphi_\tau x) - f(\varphi_\tau \psi_\tau x) = \psi(\tau,\tau) - \varphi(\tau,\tau) =$$

$$= \left(\psi - \varphi + \left(\frac{\partial \psi}{\partial \sigma} + \frac{\partial \psi}{\partial \rho} - \frac{\partial \varphi}{\partial \rho} - \frac{\partial \varphi}{\partial \sigma}\right)\tau + \left(\frac{1}{2}\frac{\partial^2 \psi}{\partial \sigma^2} + \frac{1}{2}\frac{\partial^2 \psi}{\partial \rho^2} + \frac{\partial^2 \psi}{\partial \rho \partial \sigma} - \frac{1}{2}\frac{\partial^2 \varphi}{\partial \rho^2} - \frac{1}{2}\frac{\partial^2 \varphi}{\partial \sigma^2} - \frac{\partial^2 \varphi}{\partial \sigma \partial \rho}\right)\tau^2 + \ldots\right)\Big|_{0,0}$$

$$= \tau^2\left(\xi \cdot d(\zeta \cdot df) - \zeta \cdot d(\xi \cdot df)\right)\Big|_x =$$

$$= \tau^2 [\xi, \zeta] \cdot df\,\big|_x$$